Math 115A A, Lecture 2
Real Analysis

Sample Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

(a) [5pts.] State the dimension theorem. Be sure to include all relevant hypotheses.

> **Solution:** Let V be a finite-dimensional vector space, and $T : V \to W$ be a linear transformation. Then $\dim(V) = \text{nullity}(T) + \text{rank}(T)$.

(b) [5pts.] Starting from the dimension theorem, give an argument that if $T : V \to W$ is a linear map between two n-dimensional vector spaces, then the T is onto if and only if T is one-to-one.

> **Solution:** We know that T is onto $\Leftrightarrow R(T) = W \Leftrightarrow \text{rank}(T) = \dim(W) = n$. However, since $\dim(V) = n$, by the dimension theorem, $\text{rank}(T) = n \Leftrightarrow \text{nullity}(T) = 0 \Leftrightarrow N(T) = \{0\}$. Since $N(T) = \{0\}$ if and only if T is one-to-one, we are done.

Problem 2.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the x-axis along the line $y = 2x$.

(a) [5pts.] Give a basis of eigenvectors of T, with corresponding eigenvalues.

> **Solution:** We observe that since T is a projection onto the x-axis, T maps any vector on the x-axis to itself. Ergo if $v = (1, 0)$, then $T(v) = (1, 0) = 1(1, 0)$, so $(1, 0)$ is an eigenvector with eigenvalue $\lambda = 1$. Furthermore, since we project along the line $y = 2x$, T maps any vector along the line to zero. Hence $(1, 2)$ is an eigenvector of T with eigenvalue $\lambda = 0$.

(b) [5pts.] Find the matrix of T in the standard basis for $\mathbb{R}^2$.

> **Solution:** Let β be the standard basis for $\mathbb{R}^2$ and β' be the basis $\{(1, 0), (1, 2)\}$. Then from part (a), we observe that
>
> $$[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
>
> Now, if Q is the change of coordinates matrix $[I_v]_{\beta}^{\beta'}$, we know that $[T]_\beta = Q^{-1}[T]_{\beta'}Q$. In fact, it's easiest to compute $Q^{-1} = [I_v]_{\beta'}^{\beta}$, which has columns $(1, 0)$ and $(1, 2)$ (the elements of β' written in terms of β). In other words,
>
> $$Q^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$
>
> It follows that
>
> $$Q = \begin{pmatrix} 1 & \frac{-1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

So we finally compute

$$[T]_\beta = Q^{-1}[T]_{\beta'}Q = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & \frac{-1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & \frac{-1}{2} \\ 0 & 0 \end{pmatrix}$$

Problem 3.

Let $T : V \to V$ be a linear transformation from a finite-dimensional vector space to itself.

(a) [5pts.] Prove that $T^2 = T_0$ if and only if $R(T) \subset N(T)$.

Solution: First, suppose that $T^2 = T_0$. This implies that for all $v \in V$, $T(T(v)) = 0$. In particular we see that $T(v) \in N(T)$ for all v, implying that $R(T) \subset N(T)$. Conversely, suppose $R(T) \subset N(T)$. Then for all $v \in V$, since $T(v) \in R(T) \subset N(T)$, we must ahve $T(T(v)) = 0$, or equivalently $T^2(v) = 0$. Ergo T^2 is the zero transformation T_0.

(b) [5pts.] Let β be a basis for V. If $T^k = T_0$ for some k, what is $\det([T]_\beta)$?

Solution: We know that $[T^k]_\beta = 0$, since the zero transformation is represented by the zero matrix in any basis. But $[T^k]_\beta = [T]_\beta^k$, so $[T]_\beta^k = 0$. Ergo $\det([T]_\beta^k) = \det(0) = 0$. But since the determinant of a product of matrices is the product of the determinants, $0 = \det([T]_\beta^k) = \det([T]_\beta)^k$, implying that $\det([T]_\beta) = 0$. (An alternate proof can also be given by showing that $\operatorname{rank}(T)$ must be less than $n = \dim(V)$.)

Problem 4.

(a) [5pts.] What does it mean for two vector spaces V and W to be isomorphic?

Solution: We say that two vector spaces V and W are isomorphic if there is an invertible linear transformation $T : V \to W$.

(b) [5pts.] Let V and W be finite dimensional vector spaces. Prove that a linear transformation $T : V \to W$ is an isomorphism if and only if it maps any basis β for V to a basis $T(\beta)$ for W.

Solution: Recall that in general, if $\beta = \{v_1, \cdots, v_n\}$ is a basis for V, $T(\beta) = \{T(v_1), \cdots, T(v_n)\}$ is a generating set for $R(T)$. Now, first suppose that T is an isomorphism. Then T is onto, so $R(T) = W$, implying that $T(\beta)$ is a generating set for W. Furthermore, we claim that $T(\beta)$ is linearly independent. For if $a_1T(v_1) + \cdots + a_nT(v_n) = 0$, then by linearity $T(a_1v_1 + \cdots a_nv_n) = 0$.

Since T is one-to-one, this implies $a_1v_1 + \cdot + a_nv_n = 0$, so since β is a basis, $a_i = 0$ for $1 \leq i \leq n$. Ergo $T(\beta)$ is a linearly independent generating set for W, hence a basis.

Now suppose that T maps any basis β for V to a basis $T(\beta)$ for W. Then we see that $\dim(V) = \dim(W)$, since β and $T(\beta)$ have the same number of elements, and furthermore since $T(\beta)$ is a generating set for $R(T)$ and $T(\beta)$ is a basis for W, $R(T) = W$. Ergo T is onto, so since $\dim(V) = \dim(W)$, T is in fact an isomorphism.

Problem 5.

(a) [5pts.] Let $A = \lambda I_n$ be a diagonal $n \times n$ matrix all of whose diagonal entries are λ. Prove that if B is an $n \times n$ matrix similar to A, then $B = A$.

Solution: Suppose $B = QAQ^{-1}$. Multiplying Q^{-1} by the matrix $A = \lambda I_n$ has the effect of multiplying every entry in Q^{-1} by λ. Ergo we have

$$B = QAQ^{-1} = Q(\lambda Q^{-1}) = \lambda(QQ^{-1}) = \lambda I_n.$$

So A is only similar to itself.

(b) [5pts.] Prove that the transformation

$$\begin{aligned} T : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (x + y, y) \end{aligned}$$

is not diagonalizable.

Solution: We observe that the matrix of T with respect to the standard basis β for $\mathbb{R}^2$ is

$$[T]_\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The characteristic polynomial of this matrix is $f(t) = (1 - t)^2$, and the single eigenvalue of the matrix, hence also of the transformation T, is $\lambda = 1$. If T is diagonalizable, then it must be represented by the diagonal matrix $1 \cdot I_2 = I_2$. But I_2 is similar only to itself, by part (a), and in particular is not similar to the matrix $[T]_\beta$ above. Ergo these matrices cannot represent the same transformation.